

Bohmian trajectories for photons

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The first examples of Bohmian trajectories for photons have been worked out for simple situations, using the Kemmer-Duffin-Harishchandra formalism.

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I. INTRODUCTION

It is generally believed that only massive fermions have Bohmian trajectories but bosons do not. This is usually attributed to the impossibility of constructing a relativistic quantum mechanics of bosons with a conserved four-vector probability current density with a positive definite time component. However, it has now been shown [1] that a consistent relativistic quantum mechanics of spin 0 and spin 1 bosons can be developed using the Kemmer equation [2]

$$(i \hbar \beta_\mu \partial^\mu + m_0 c) \psi = 0 \quad (1)$$

where the matrices β satisfy the algebra

$$\beta_\mu \beta_\nu \beta_\lambda + \beta_\lambda \beta_\nu \beta_\mu = \beta_\mu g_{\nu\lambda} + \beta_\lambda g_{\nu\mu}. \quad (2)$$

The 5×5 dimensional representation of these matrices describes spin 0 bosons and the 10×10 dimensional representation describes spin 1 bosons. The fact that a conserved four-vector current with a positive definite time component can be defined using this formalism can be seen as follows. Multiplying (1) by β_0 , one obtains the Schrödinger form of the equation

$$i \hbar \frac{\partial \psi}{\partial t} = [-i \hbar c \tilde{\beta}_i \partial_i - m_0 c^2 \beta_0] \psi \quad (3)$$

where $\tilde{\beta}_i \equiv \beta_0 \beta_i - \beta_i \beta_0$. Multiplying (1) by $1 - \beta_0^2$, one obtains the first class constraint

$$i \hbar \beta_i \beta_0^2 \partial_i \psi = -m_0 c (1 - \beta_0^2) \psi. \quad (4)$$

It implies the four conditions $\vec{A} = \vec{\nabla} \times \vec{B}$ and $\vec{\nabla} \cdot \vec{E} = 0$ in the spin-1 case. The reader is referred to Ref. [1] for further discussions regarding the significance of this constraint.

If one multiplies equation (3) by ψ^\dagger from the left, its hermitian conjugate by ψ from the right and adds the resultant equations, one obtains the continuity equation

$$\frac{\partial (\psi^\dagger \psi)}{\partial t} + \partial_i \psi^\dagger \tilde{\beta}_i \psi = 0. \quad (5)$$

This can be written in the form

$$\partial^\mu s_\mu = 0 \quad (6)$$

where $s_\mu = -\Theta_{\mu\nu} a^\nu$ (with $a^\nu a_\nu = 1$ where a^ν is the unit four-velocity of the observer), $\Theta_{\mu\nu} = -m_0 c^2 \bar{\psi} (\beta_\mu \beta_\nu + \beta_\nu \beta_\mu - g_{\mu\nu}) \psi$ is the symmetric energy-momentum tensor so that $\Theta_{00} = -m_0 c^2 \psi^\dagger \psi < 0$. Notice that $s_\mu s^\mu = \Theta_{\mu\nu} \Theta^{\mu\nu} \geq 0$, so that s_μ is time-like. Thus, it is possible to define a wave function $\phi = \sqrt{m_0 c^2 / E} \psi$ (with $E = -\int \Theta_{00} dV$) such that $\phi^\dagger \phi$ is non-negative and normalized and can be interpreted as a probability density. The conserved probability current density is $s_\mu = -\Theta_{\mu 0} / E = (\phi^\dagger \phi, -\phi^\dagger \tilde{\beta}_i \phi)$ [1].

Notice that according to the equation of motion (3), the velocity operator for massive bosons is $c \tilde{\beta}_i$, so that the Bohmian 3-velocity can be defined by

$$v_i = \frac{dx_i}{dt} = \gamma^{-1} u_i = c \frac{u_i}{u_0} = c \frac{s_i}{s_0} = c \frac{\psi^\dagger \tilde{\beta}_i \psi}{\psi^\dagger \psi}. \quad (7)$$

It follows from equation (3) that $c\tilde{\beta}_i$ is the velocity operator whose eigenvalues are $\pm c$. Therefore, $v_\mu v^\mu = 0$, and so the Bohmian velocity is always timelike. Integrating equation (7), one obtains a system of Bohmian trajectories $x_i(t)$ corresponding to different initial positions of the particle. In Bohmian mechanics one assumes that the initial distribution of the positions is given by $|\psi(0)|^2$. The continuity equation (5) then guarantees that the distribution will agree with quantum mechanics at all future times. The (Gibbs) ensemble averages of all dynamical variables in Bohmian mechanics will therefore always agree with the expectation values of the corresponding hermitian operators in quantum mechanics.

The theory of massless spin 0 and spin 1 bosons cannot be obtained simply by taking the limit m_0 going to zero. One has to start with the equation [3]

$$i \hbar \beta_\mu \partial^\mu \psi + m_0 c \Gamma \psi = 0 \quad (8)$$

where Γ is a matrix that satisfies the following conditions:

$$\Gamma^2 = \Gamma \quad (9)$$

$$\Gamma \beta_\mu + \beta_\mu \Gamma = \beta_\mu. \quad (10)$$

Multiplying (8) from the left by $1 - \Gamma$, one obtains

$$\beta_\mu \partial^\mu (\Gamma \psi) = 0. \quad (11)$$

Multiplying (8) from the left by $\partial_\lambda \beta^\lambda \beta^\nu$, one also obtains

$$\partial^\lambda \beta_\lambda \beta_\nu (\Gamma \psi) = \partial_\nu (\Gamma \psi). \quad (12)$$

It follows from (11) and (12) that

$$\square (\Gamma \psi) = 0 \quad (13)$$

which shows that $\Gamma \psi$ describes massless bosons. The Schrödinger form of the equation

$$i \hbar \frac{\partial (\Gamma \psi)}{\partial t} = -i \hbar c \tilde{\beta}_i \partial_i (\Gamma \psi) \quad (14)$$

and the associated first class constraint

$$i \hbar \beta_i \beta_0^2 \partial_i \psi + m_0 c (1 - \beta_0^2) \Gamma \psi = 0 \quad (15)$$

follow by multiplying (8) by β_0 and $1 - \beta_0^2$ respectively. Equation (14) implies the Maxwell equations $\text{curl} \vec{E} = -(\mu/c) \partial_t \vec{H}$ and $\text{curl} \vec{H} = (\epsilon/c) \partial_t \vec{E}$ if

$$\Gamma \psi^T = (1/\sqrt{m_0 c^2})(-D_x, -D_y, -D_z, B_x, B_y, B_z, 0, 0, 0) \quad (16)$$

The constraint (15) implies the relations $\text{div} \vec{E} = 0$ and $\vec{B} = \text{curl} \vec{A}$. The symmetrical energy-momentum tensor is

$$\Theta_{\mu\nu} = -\frac{m_0 c^2}{2} \bar{\psi} (\beta_\mu \beta_\nu + \beta_\nu \beta_\mu - g_{\mu\nu}) \Gamma \psi \quad (17)$$

and so the energy density

$$\mathcal{E} = -\Theta_{00} = \frac{m_0 c^2}{2} \psi^\dagger \Gamma \psi = \frac{1}{2} [\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B}] \quad (18)$$

is positive definite. The rest of the arguments are analogous to the massive case.

The Bohmian 3-velocity v_i for massless bosons can be defined by

$$v_i = c \frac{\psi^T \Gamma \tilde{\beta}_i \Gamma \psi}{\psi^T \Gamma \psi} \quad (19)$$

Using arguments similar to the case of massive bosons, it is easy to see that the Bohmian velocities for massless bosons are also timelike. Integrating equation (19) with different initial positions, one gets a system of Bohmian trajectories

for the photon. Neutral massless vector bosons are very special in quantum mechanics. Their wave function is real, and so their charge current $j_\mu = \phi^T \beta_\mu \phi$ vanishes. However, their probability current density s_μ does not vanish. Furthermore, s_i turns out to be proportional to the Poynting vector, as it should.

In this paper we compute Bohmian trajectories for photons for certain simple but interesting cases. Integral curves of the Poynting vector for localized wave packets in classical electrodynamics were first plotted by Prosser [4]. They are lines of energy flow in classical electrodynamics and cannot be interpreted as particle trajectories. A particle trajectory interpretation of these curves is possible only within the context of a proper relativistic quantum mechanics of indivisible photons. This is what we have done to calculate Bohmian velocities and hence trajectories for photons, carrying the entire interpretational package of Bohmian mechanics. It is only incidental that such trajectories happen to coincide with the integral curves of the Poynting vector for single photons. However, in the case of two photons, the Bohmian trajectories are computed from a two photon symmetrized wave function which has no classical analogue. In this sense, the Bohmian trajectories calculated in the following sections represent the first plots of photon trajectories.

The plan of the paper is as follows. In Section II we study the trajectories in Young's double-slit experiment. In section III, we compute the trajectories corresponding to two down-converted photons passing through a double-slit. In Section IV we plot the Bohmian trajectories for reflection and refraction through a glass slab. We make some concluding remarks in Section V.

II. SINGLE PHOTON DOUBLE-SLIT INTERFERENCE

Let us now consider the specific case of double-slit interference of single photons. If the slits A and B have a non-zero width d significantly larger than the de Broglie wavelength of the particles ($d \gg \lambda$), the slits will convert plane incident waves into plane diffracted waves sufficiently far from them (the case of Fraunhofer diffraction). One can see this by carrying out the necessary approximations [5] on the single-particle spherical wave at a point P , arriving from a point within a slit at a distance $x = \pm \xi$ from the origin, and integrating over the slit [6]. The wave function at a point (x, y) at a sufficient distance $D \gg d^2/\lambda$ to the right of the plane of the slits is given by¹

$$\psi(x, y) = \mathcal{M}_A g_A \frac{\exp(ikr_A)}{r_A} + \mathcal{M}_B g_B \frac{\exp(ikr_B)}{r_B} \quad (20)$$

where g_A and g_B are the diffraction factors given by

$$g_{A,B} = \frac{\sin(kyd/2D)}{kyd/2D} \quad (21)$$

and \mathcal{M}_A and \mathcal{M}_B are the Kemmer-Duffin wave functions given by

$$\mathcal{M}_A = \begin{pmatrix} -E_0 \sin(\theta_A) \\ -E_0 \cos(\theta_A) \\ 0 \\ 0 \\ 0 \\ B_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (22)$$

and

¹Henceforth we shall write ψ in place of $\Gamma\psi$ for brevity of notation.

$$\mathcal{M}_B = \begin{pmatrix} E_0 \sin(\theta_B) \\ -E_0 \cos(\theta_B) \\ 0 \\ 0 \\ 0 \\ B_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (23)$$

where θ_A and θ_B are the angles of diffraction from slits A and B respectively.

Using the above wave function the components for Bohmian velocity are given by

$$\begin{aligned} v_x &= \frac{2E_0 B_0}{\psi^\dagger \psi} \left(g_A^2 \cos(\theta_A) + g_B^2 \cos(\theta_B) + g_A g_B \cos[k(r_A - r_B)] (\cos(\theta_A) + \cos(\theta_B)) \right) \\ v_y &= \frac{2E_0 B_0}{\psi^\dagger \psi} \left(-g_A^2 \sin(\theta_A) + g_B^2 \sin(\theta_B) + g_A g_B \cos[k(r_A - r_B)] (\sin(\theta_B) - \sin(\theta_A)) \right) \end{aligned} \quad (24)$$

with $\psi^\dagger \psi$ given by

$$\psi^\dagger \psi = (E_0^2 + B_0^2)(g_A^2 + g_B^2) + 2g_A g_B \left(E_0^2 \cos(\theta_A + \theta_B) + B_0^2 \right) \cos[k(r_A - r_B)] \quad (25)$$

The Bohmian trajectories for photons can now be plotted for different initial positions along the slits using the above expressions for the velocity. We have taken a uniform distribution of the initial positions for both the slits. (See Figure 1). The trajectories clearly correspond to the probability density obtained from standard quantum theory at any line parallel to the line joining the slits (y -axis). The trajectories are similar to the trajectories of massive particles [7].

III. BOHMIAN TRAJECTORIES OF A PAIR OF DOWN-CONVERTED PHOTONS

Before we proceed to compute the Bohmian trajectories for a pair of photons, the following point needs to be clarified. Let us define the rank-2 tensor current

$$s_{\mu\nu}(x_1, x_2) = c \bar{\psi}(x_1, x_2) \left(\beta_\mu^{(1)} \beta_\lambda^{(1)} + \beta_\lambda^{(1)} \beta_\mu^{(1)} - g_{\mu\lambda} \right) a^\lambda \left(\beta_\nu^{(2)} \beta_\rho^{(2)} + \beta_\rho^{(2)} \beta_\nu^{(2)} - g_{\nu\rho} \right) a^\rho \Gamma \psi(x_1, x_2) \quad (26)$$

for wave functions which satisfy the symmetry $\psi(x_1, x_2) = \psi(x_2, x_1)$. Then the i -th component of the Bohmian velocity for the n -th particle ($n = 1, 2$) is

$$v_i^{(n)}(x_1, x_2) = c \frac{s_{i0}(x_1, x_2)}{s_{00}(x_1, x_2)} \quad (27)$$

Using similar arguments to those presented in section I, it is clear that this Bohmian velocity is also time-like. The expression (27) however appears to be non-covariant because the two sides transform differently. Nevertheless, it is possible to write it in a manifestly covariant form by introducing a foliation of spacetime with spacelike hypersurfaces Σ with future oriented unit normals $\eta^\mu(x)$ at every point x of Σ such that $\eta^\mu(x)\eta_\mu(x) = 1$. Then

$$\begin{aligned} v_i^{(1)}(x_1, x_2) &= c \frac{s_{i\mu}(x_1, x_2) \eta^\mu(x_2)}{s_{\mu\nu}(x_1, x_2) \eta^\mu(x_1) \eta^\nu(x_2)} \\ v_i^{(2)}(x_1, x_2) &= c \frac{s_{i\mu}(x_1, x_2) \eta^\mu(x_1)}{s_{\mu\nu}(x_1, x_2) \eta^\mu(x_1) \eta^\nu(x_2)} \end{aligned} \quad (28)$$

The fact that EPR entangled states can be written in a manifestly covariant form using this technique of spacetime foliation was first shown by Ghose and Home [8]. The same technique was used by Durr et al. [9] and Holland [10] in

the context of Bohmian velocities for multiparticle entangled states to demonstrate their relativistic covariance. For further details, see [9,10].

We now consider an experiment in which a pair of down converted photons is made to pass through two identical slits. We will compute the Bohmian trajectories for this case in the limit of Fraunhofer diffraction. The two-particle wave function (in the Fraunhofer limit, i.e., $x_1 = x_2 = D \gg d^2/\lambda$) is given by

$$\psi(y_1, y_2) = \frac{\exp(2ikD)}{D^2} d^2 g_1 g_2 \left(\mathcal{M}_{A1} \mathcal{M}_{B2} \exp(-ika(y_1 - y_2)/D) + \mathcal{M}_{A2} \mathcal{M}_{B1} \exp(ika(y_1 - y_2)/D) \right) \quad (29)$$

After substituting the expressions for the Kemmer-Duffin matrix elements and the diffraction factors, we obtain the following expressions for the Bohmian velocities of the two photons:

$$v_{1x} = \frac{c}{\psi^\dagger \psi} \left(-2 \left(g_1^4 \cos(\theta_A) + g_2^4 \cos(\theta_B) \right) + g_1^2 g_2^2 \left(1 + \cos(\theta_A + \theta_B) \right) \left(\cos(\theta_A) + \cos(\theta_B) \right) \right) \quad (30)$$

$$v_{2x} = v_{1x} \quad (31)$$

$$v_{1y} = \frac{c}{\psi^\dagger \psi} \left(-2 \left(g_1^4 \sin(\theta_A) - g_2^4 \sin(\theta_B) \right) + g_1^2 g_2^2 \left(1 + \cos(\theta_A + \theta_B) \right) \left(-\sin(\theta_A) + \sin(\theta_B) \right) \right) \quad (32)$$

$$v_{2y} = -v_{1y} \quad (33)$$

where $\psi^\dagger \psi$ is given by

$$\psi^\dagger \psi = \frac{8d^4 g_1^2 g_2^2 E_0^2 B_0^2}{D^4} \left[1 + \frac{(\cos(\theta_A + \theta_B) + 1)^2}{4} \cos(2ka(y_1 - y_2)/D) \right] \quad (34)$$

The second cosine term represents a fourth order interference in the joint detection probability of the two photons [11].

The Bohmian trajectories are plotted in Figure 2. It can be checked that they agree with the joint detection probability amplitude obtained on a plane parallel to the plane of the slits. Again, they are similar to the trajectories one obtains for the symmetrized wave function of two massive particles [7]. Note that the trajectories are symmetric about the x-axis, and the trajectories in the upper and lower half-planes do not cross.

IV. REFLECTION AND REFRACTION THROUGH A GLASS SLAB

Finally, let us consider the example of refraction of light through a glass slab. We consider both the air-glass interfaces separately and combine the solutions. To obtain the solution of the Kemmer-Duffin equation in this case, we must first solve Maxwell's equations for this case. Let the electric field be polarized along the y direction and let the wave propagate along the x direction. The air-glass interface is taken at $x = 0$. Let the amplitude of the electric field be represented by a gaussian wave packet² centered at x_0 , i.e.,

$$\begin{aligned} E_x &= E_0 \exp\left(\frac{-(x - ct - x_0)^2}{2\sigma_0}\right) \\ E_y &= 0 \\ E_z &= 0 \end{aligned} \quad (35)$$

Taking into account the boundary conditions at the air-glass interface, one obtains

²Such wave packets are nowadays routinely produced in the laboratory in down-conversion experiments. See, for example [12].

$$E_x(x, t) = \begin{pmatrix} \left(E_0 \exp\left(\frac{-(x-ct-x_0)^2}{2\sigma_0}\right) + \frac{1-n}{1+n} \left(E_0 \exp\left(\frac{-(x+ct-x_0)^2}{2\sigma_0}\right) \right) \right), & x < 0 \\ \left(\frac{2}{1+n} E_0 \exp\left(\frac{-(nx-ct-x_0)^2}{2\sigma_0}\right) \right), & x \geq 0 \end{pmatrix} \quad (36)$$

where n is the refractive index. The corresponding magnetic field is given by

$$B_z(x, t) = \begin{pmatrix} \left(\frac{E_0}{c} \exp\left(\frac{-(x-ct-x_0)^2}{2\sigma_0}\right) - \frac{1-n}{1+n} \left(\frac{E_0}{c} \exp\left(\frac{-(x+ct-x_0)^2}{2\sigma_0}\right) \right) \right), & x < 0 \\ \left(\frac{2}{1+n} \frac{E_0}{c} \exp\left(\frac{-(nx-ct-x_0)^2}{2\sigma_0}\right) \right), & x \geq 0 \end{pmatrix} \quad (37)$$

The Kemmer-Duffin-Harish Chandra wave function is therefore given by

$$\psi = \begin{pmatrix} -E_x \\ -E_y \\ -E_z \\ B_x \\ B_y \\ B_z \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (38)$$

Using these expressions for the electric and magnetic fields, one obtains the Bohmian velocity to be

$$v_x = \begin{pmatrix} c \left(\frac{\exp\left(\frac{-(x-ct-x_0)^2}{\sigma_0}\right) - \frac{(1-n)^2}{(1+n)^2} \left(E_0 \exp\left(\frac{-(x+ct+x_0)^2}{\sigma_0}\right) \right)}{\exp\left(\frac{-(x-ct-x_0)^2}{\sigma_0}\right) + \frac{(1-n)^2}{(1+n)^2} \left(E_0 \exp\left(\frac{-(x+ct-x_0)^2}{2\sigma_0}\right) \right)} \right), & x < 0 \\ \left(\frac{c}{n} \right), & x \geq 0 \end{pmatrix} \quad (39)$$

Similarly, the solutions for the electric and magnetic fields, and the corresponding expressions for Bohmian velocity can be obtained for reflection and refraction at the next glass-air interface placed at $x = 0.2$. These two sets of solutions are combined to obtain the trajectories for photons reflected and transmitted through the glass slab. The trajectories for a particular set of initial positions are plotted in Figure 3.

V. CONCLUSIONS

Bohm and his coworkers have all along emphasized a fundamental difference between fermions and bosons in that fermions, in their view, are particles, whereas bosons are fields. This asymmetry in the Bohmian picture of fermions and bosons arose due to the absence, in their view, of a consistent relativistic quantum mechanics of bosons with a conserved four-vector current which is time-like and whose time component is positive. Such a formulation was provided by Ghose et al. [1,8] and it was shown that Bohmian trajectories for relativistic bosons could be defined [13]. Just as the actual plotting of Bohmian trajectories for nonrelativistic particles was an important advance [14], it is equally important to demonstrate the actual nature of Bohmian trajectories for relativistic bosons in simple physical situations, particularly because such trajectories were thought not to exist by Bohm himself. This does not in any way detract from the significance of Bohm's general point of view regarding the causal interpretation. In our view these trajectories constitute a significant support of Bohm's causal interpretation by removing an unnecessary asymmetry between fermions and bosons from it. In case there is any truth in supersymmetry, such an asymmetry would be fatal for Bohmian mechanics.

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Figure Captions:

Figure 1. Bohmian trajectories of photons for self-interference through a pair of identical slits centered at $y = -0.0002$ and $y = 0.0002$.

Figure 2. Bohmian trajectories for a pair of photons passing through two identical slits. Note that there is no crossing of trajectories between the upper and lower half planes.

Figure 3. Bohmian trajectories for photons passing through a glass slab placed at $0 \leq x \leq 0.2$. Reflection and refraction are seen at both the air-glass interfaces.





